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A sharp threshold of blow-up for coupled nonlinear Schrödinger equations

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Abstract
This paper is concerned with the coupled supercritical nonlinear Schrödinger equations which have applications in many physical problems, especially in nonlinear optics. Two types of new invariant evolution flows are established. A sharp threshold of blow-up and global existence of solutions for the equations is derived. It is shown that the main result obtained includes parts of those presented by Fanelli and Montefusco (2007 *J. Phys. A: Math. Theor.* **40** 14139–50).

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1. Introduction

In this paper we consider the coupled supercritical nonlinear Schrödinger equations

$$\begin{cases} \mathrm{i}\phi_t + \triangle\phi + (|\phi|^{2p} + \beta|\phi|^{p-1}|\psi|^{p+1})\phi = 0, \\ \mathrm{i}\psi_t + \triangle\psi + (|\psi|^{2p} + \beta|\psi|^{p-1}|\phi|^{p+1})\psi = 0, \end{cases} \tag{1.1}$$

with initial data

$$\phi(0,x) = \phi_0(x), \qquad \psi(0,x) = \psi_0(x), \tag{1.2}$$

where $\phi, \psi : R \times R^n \to C$, $\phi_0, \psi_0 : R^n \to C$, n is the space dimension, $\mathrm{i} = \sqrt{-1}$, $\triangle$ is the Laplace operator on R^n and β is a real positive constant. $0 < p < A_n$ in which $A_n = \infty$ if $n = 1, 2$ and $A_n = \frac{2}{(n-2)}$ if $n \geqslant 3$.

The system (1.1) has several applications in physics, especially in nonlinear optics (see [2, 7, 10, 15, 22]). One of the most important applications is as a mathematical model for propagation of polarized laser beams in a birefringer Kerr medium in nonlinear optics [7], and for such a problem, ϕ and ψ represent respectively the components of the slowly varying envelope of the electrical field, x denotes the orthogonal variables and t is the distance in the direction of propagation. If $n = 1$, the system (1.1) corresponds to the propagation in a planar geometry. The case $n = 2$ is the propagation in a bulk medium and the case $n = 3$ is the propagation of pulses in a bulk medium with time dispersion.

The focusing nonlinear terms in (1.1) describe the dependence of the refraction index of the material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ is the birefringence intensity and describes the coupling between the two components of the electric-field envelope. The case $p = 1$ (i.e. cubic nonlinearities in (1.1)) is known as Kerr nonlinearity in the physical literature.

For the Cauchy problem (1.1) and (1.2), the local well-posedness described by the following proposition was established in the energy space $H^1(R^n) \times H^1(R^n)$ (see [3] or [17]).

Proposition 1 (Local existence)**.** *Assume that* $0 < p < A_n$. *Then for any* $(\phi_0, \psi_0) \in H^1(R^n) \times H^1(R^n)$ *there exist* $T > 0$ *and a unique solution* $(\phi, \psi) \in C([0, T); H^1(R^n) \times H^1(R^n))$ *such that either* $T = \infty$ *or* $T < \infty$ *and* $\lim_{t\to T} \int_{R^n} (|\nabla\phi|^2 + |\nabla\psi|^2) = \infty$. *Moreover, the system (1.1) admits the mass and the energy conservation in the space* $H^1(R^n) \times H^1(R^n)$. *Namely, mass (L^2 norm)*

$$M[\phi(t), \psi(t)] := \|\phi\|_2^2 + \|\psi\|_2^2 = M[\phi_0, \psi_0] \tag{1.3}$$

and energy

$$\begin{aligned} E[\phi, \psi] &:= \frac{1}{2}\left(\|\nabla\phi(t)\|_2^2 + \|\nabla\psi(t)\|_2^2\right) - \frac{1}{2p+2}\left(\|\phi\|_{2p+2}^{2p+2} + 2\beta\|\phi\psi\|_{p+1}^{p+1} + \|\psi\|_{2p+2}^{2p+2}\right) \\ &= E[\phi_0, \psi_0]. \end{aligned} \tag{1.4}$$

Here and hereafter, for simplicity, we write $\|\cdot\|_p$ for the norm in the space $L^p(R^n)$.

Recently, much attention was paid to study the blow-up and global existence of the Cauchy problem (1.1) and (1.2) (see, for example, [6, 7, 13, 15]), and the following results have been established.

- (i) When $0 < p < \frac{2}{n}$, the solutions of the Cauchy problem (1.1) and (1.2) exist globally in time (see [7]).
- (ii) When $p \geqslant \frac{2}{n}$, the solutions of the Cauchy problem (1.1) and (1.2) blow up in a finite time for some initial data ($E[\phi_0, \psi_0] < 0$), especially for a class of sufficiently large data (see [6, 7, 13, 15]). On the other hand, the solutions of the Cauchy problem (1.1) and (1.2) globally exist for other initial data, especially for a class of sufficiently small data (see [3, 7, 13]).

Obviously $p = \frac{2}{n}$ is a critical value and called the critical nonlinear power exponent, while $p > \frac{2}{n}$ is called the supercritical nonlinear power exponent. A natural question arises for $p \geqslant \frac{2}{n}$, that is, whether a sharp threshold can be found for the initial data which separate blow-up and global existence. For the single Schrödinger equation, this problem has been extensively studied (see [4, 5, 8, 9, 19–21]) and it is suggested that the sharp threshold of blow-up and global existence is related to the solution of a corresponding stationary equation. For the system (1.1), various attempts have also been made to study the blow-up threshold. To study the blow-up threshold, the following stationary system associated with (1.1) is considered:

$$\begin{cases} \triangle Q - \dfrac{(2-n)p+2}{2} Q + (|Q|^{2p} + \beta|Q|^{p-1}|R|^{p+1})Q = 0, \\ \triangle R - \dfrac{(2-n)p+2}{2} R + (|R|^{2p} + \beta|R|^{p-1}|Q|^{p+1})R = 0. \end{cases} \tag{1.5}$$

We note that $(e^{i\omega t}Q(x), e^{i\omega t}R(x))$ with $\omega = \frac{(2-n)p+2}{2}$ is a solitary wave solution of the system (1.1). For the elliptic system of this kind, many authors have studied the existence of positive solutions (see, for example, [1, 12, 14, 16]). Maia, Montefusco and Pellacci [14] obtained the existence of some qualitative properties of the ground-state solutions. Using the existence of the ground-state solutions and an argument similar to [19], Fanelli and Montefusco [7] established the sharp Gagliardo–Nirenberg inequality. With this inequality the authors further derived the blow-up threshold of the system (1.1) with $p = \frac{2}{n}$(critical case) in terms of the ground state. More specifically, Fanelli and Montefusco [7] proved that if $\|\phi_0\|_2^2 + \|\psi_0\|_2^2 < \|Q\|_2^2 + \|R\|_2^2$, the solution of the Cauchy problem (1.1) and (1.2) exists for all time. Moreover, they constructed the blow-up solution with $\|\phi_0\|_2^2+\|\psi_0\|_2^2 = \|Q\|_2^2+\|R\|_2^2$.

For the critical and supercritical case $p \geqslant \frac{2}{n}$, Ma and Zhao [13] established a sharp threshold of blow-up and global existence in a subset of the energy space $H^1(R^n) \times H^1(R^n)$ by using the cross-constrained variational argument. However, it is not exactly known how large the subset is. The essential difficulty is that their results are not explicit and cannot be precisely computed.

In this paper, we consider the critical and supercritical case $p \geqslant \frac{2}{n}$ of the Cauchy problem (1.1) and (1.2). By improving the arguments of Fanelli and Montefusco [7] and Ma and Zhao [13] and further exploiting the Hamiltonian invariants, applying the sharp Gagliardo–Nirenberg inequality, we get a new blow-up threshold in terms of the ground state. In comparison with the results in [7] and [13], our results are explicit and can be precisely computed. In addition, our main result is more general and includes parts of those presented by Fanelli and Montefusco [7] as a special case.

The rest of the paper is organized as follows. In section 2, we summarize the variational characterization of the ground state. In section 3, we recall the Gagliardo–Nirrenberg inequality and define several constants which are fundamental in proofs of theorems 1 and 2. Section 4 deals with the invariant set. Section 5 is devoted to the study of the blow-up threshold, and finally section 6 states some concluding remarks.

2. Variational characterization of the ground state

In this section, we consider the following system:

$$\begin{cases} a\triangle u - bu + c(|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0, \\ a\triangle v - bv + c(|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0, \end{cases} \tag{2.1}$$

with $a, b, c > 0$ and a related minimization problem

$$\alpha := \inf_{u,v\in H^1(R^n)} J_{n,p,\beta}(u, v), \tag{2.2}$$

where

$$J_{n,p,\beta}(u, v) = \frac{\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{pn/2}\left(\|u\|_2^2 + \|v\|_2^2\right)^{p+1-pn/2}}{\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}}, \quad u, v \in H^1(R^n).$$

For the system (2.1), many authors have studied the existence of positive solutions (see [1, 12, 14, 16]). Particularly, Fanelli and Montefusco [7] obtained the existence result and some qualitative properties of the ground-state solutions for the system (2.1).

Definition 1. *Let $\mathcal{X}$ be the set of the solutions of (2.1), namely*

$$\mathcal{X} := \{(u, v) \in H^1(R^n) \times H^1(R^n), (u, v) \neq (0, 0), (u, v) \text{ solves system } (2.1)\},$$

and let $\mathcal{G}$ be the set of the ground states of (2.1), that is,

$$\mathcal{G} := \{(U, V) \in \mathcal{X};\ I[U, V] \leqslant I[u, v], \forall (u, v) \in \mathcal{X}\},$$

where

$$I[u, v] = I_{n,p,\beta}[u, v] = \frac{a}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{b}{2}\left(\|u\|_2^2 + \|v\|_2^2\right) - \frac{c}{2p+2}\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right).$$

Various results on the properties of the ground state have been obtained [7] and [14]. In the following proposition, we summarize the vital characteristics of the ground state, including some known results and a new result.

Proposition 2. *Let $0 < p < A_n$ and $\beta > 0$; then we have*

(i) $\mathcal{G} \neq \emptyset$.
(ii) There exists $(U, V) \in \mathcal{G}$ such that $U \neq 0$ and $V \neq 0$ if and only if $\beta > 2^p - 1$.
(iii) $J_{n,p,\beta}(U, V) = \alpha$ for any $(U, V) \in \mathcal{G}$.
(iv) $\|U\|_2^2 + \|V\|_2^2 \leqslant \|u\|_2^2 + \|v\|_2^2$, for any $(u, v) \in \mathcal{X}$.

Proof. Properties (i) and (ii) were established in [14]. Property (iii) was derived in [7].

Multiplying (2.1) by (u, v) and integrating by part in R^n, we obtain

$$\begin{cases} a\|\nabla u\|_2^2 + b\|u\|_2^2 = c\left(\|u\|_{2p+2}^{2p+2} + \beta\|uv\|_{p+1}^{p+1}\right), \\ a\|\nabla v\|_2^2 + b\|v\|_2^2 = c\left(\|v\|_{2p+2}^{2p+2} + \beta\|uv\|_{p+1}^{p+1}\right), \end{cases} \tag{2.3}$$

which results in

$$a\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + b\left(\|u\|_2^2 + \|v\|_2^2\right) = c\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right). \tag{2.4}$$

Moreover, the Pohozaev identity reads

$$\frac{n-2}{2}a\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{n}{2}b\left(\|u\|_2^2 + \|v\|_2^2\right) = \frac{nc}{2p+2}\left(\|u\|_{2p+2}^{2p+2} + 2\beta\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right). \tag{2.5}$$

It follows from the definition of the ground state, (2.4) and (2.5), that (iv) holds. □

3. Gagliardo–Nirenberg inequality

Let (Q^*, R^*) be a ground-state solution of the system (2.1) with $a = \frac{pn}{2}$, $b = \frac{(2-n)p+2}{2}$ and $c = 1$, namely

$$\begin{cases} \frac{pn}{2}\triangle Q^* - \frac{(2-n)p+2}{2}Q^* + (|Q^*|^{2p} + \beta|Q^*|^{p-1}|R^*|^{p+1})Q^* = 0, \\ \frac{pn}{2}\triangle R^* - \frac{(2-n)p+2}{2}R^* + (|R^*|^{2p} + \beta|R^*|^{p-1}|Q^*|^{p+1})R^* = 0. \end{cases} \tag{3.1}$$

Equations(2.4) and (2.5) become respectively

$$\frac{pn}{2}\left(\|\nabla Q^*\|_2^2 + \|\nabla R^*\|_2^2\right) + \frac{(2-n)p+2}{2}\left(\|Q^*\|_2^2 + \|R^*\|_2^2\right) = \|Q^*\|_{2p+2}^{2p+2} + 2\beta\|Q^*R^*\|_{p+1}^{p+1} + \|R^*\|_{2p+2}^{2p+2}, \tag{3.2}$$

$$\frac{n-2}{2}\frac{pn}{2}\left(\|\nabla Q^*\|_2^2+\|\nabla R^*\|_2^2\right)+\frac{n}{2}\frac{(2-n)p+2}{2}\left(\|Q^*\|_2^2+\|R^*\|_2^2\right)$$
$$=\frac{n}{2p+2}\left(\|Q^*\|_{2p+2}^{2p+2}+2\beta\|Q^*R^*\|_{p+1}^{p+1}+\|R^*\|_{2p+2}^{2p+2}\right). \tag{3.3}$$

Putting (3.2) and (3.3) together yields

$$\|\nabla Q^*\|_2^2+\|\nabla R^*\|_2^2=\|Q^*\|_2^2+\|R^*\|_2^2. \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\|Q^*\|_{2p+2}^{2p+2}+2\beta\|Q^*R^*\|_{p+1}^{p+1}+\|R^*\|_{2p+2}^{2p+2}=(p+1)\left(\|Q^*\|_2^2+\|R^*\|_2^2\right). \tag{3.5}$$

According to part (iii) of proposition 2, we note that the minimum of (2.2) is obtained by (Q^*, R^*). From (2.2), (3.4) and (3.5), we have

$$\alpha=\frac{\left(\|Q^*\|_2^2+\|R^*\|_2^2\right)^p}{p+1}.$$

Based on the above results and using the Gagliardo–Nirenberg inequality in [7], we obtain the following lemma.

Lemma 1 (Fanelli and Montefusco [7]). *For $(\phi,\psi)\in H^1(R^n)\times H^1(R^n)$, one has*

$$\|\phi\|_{2p+2}^{2p+2}+2\beta\|\phi\psi\|_{p+1}^{p+1}+\|\psi\|_{2p+2}^{2p+2}\leqslant C_{n,p,\beta}\left(\|\phi\|_2^2+\|\psi\|_2^2\right)^{p+1-\frac{np}{2}}\left(\|\nabla\phi\|_2^2+\|\nabla\psi\|_2^2\right)^{\frac{pn}{2}} \tag{3.6}$$

with

$$C_{n,p,\beta}=\frac{1}{\alpha}=\frac{p+1}{\left(\|Q^*\|_2^2+\|R^*\|_2^2\right)^p}.$$

Set

$$Q(x)=Q^*(\lambda x),\qquad R(x)=R^*(\lambda x)$$

with $\lambda=\sqrt{\frac{pn}{2}}$; then $(Q(x), R(x))$ satisfies (1.5). Now, we define some constants and functionals that will take important roles in the subsequent sections.

Definition 2. $s_c:=\frac{n}{2}-\frac{1}{p}$, $\sigma_{p,n,\beta}:=\left(\frac{2}{pn}\right)^{\frac{1}{2p}}\sqrt{\|Q^*\|_2^2+\|R^*\|_2^2}$,
$\wedge[\phi,\psi]:=E^{s_c}[\phi,\psi]M^{1-s_c}[\phi,\psi]$,
$\mathcal{V}[\phi,\psi]:=\left(\|\nabla\phi\|_2^2+\|\nabla\psi\|_2^2\right)^{\frac{s_c}{2}}\left(\|\phi\|_2^2+\|\psi\|_2^2\right)^{\frac{1-s_c}{2}}$.

Lemma 2. *The following equalities hold:*

$$\wedge[Q,R]\equiv\left(\frac{s_c}{n}\right)^{s_c}(\sigma_{p,n,\beta})^2, \tag{3.7}$$

$$\mathcal{V}[Q,R]\equiv\sigma_{p,n,\beta} \tag{3.8}$$

and

$$C_{n,p,\beta}=\frac{p+1}{\left(\frac{np}{2}\right)\mathcal{V}^{2p}[Q,R]}. \tag{3.9}$$

Proof. In fact, we have

$$\begin{aligned}\wedge[Q,R] &= E^{s_c}[Q,R]M^{(1-s_c)}[Q,R]\\ &= \left(\frac{\lambda^2}{2}-\frac{1}{2}\right)^{s_c}\alpha^{-n}\left(\|Q^*\|_2^2+\|R^*\|_2^2\right)\\ &= \left(\frac{pn-2}{4}\right)^{s_c}\left(\frac{2}{pn}\right)^{\frac{n}{2}}\left(\|Q^*\|_2^2+\|R^*\|_2^2\right).\end{aligned}$$

On the other hand, we have

$$\left(\frac{s_c}{n}\right)^{s_c}(\sigma_{p,n,\beta})^2 = \left(\frac{pn-2}{2pn}\right)^{s_c}\left(\frac{2}{pn}\right)^{\frac{1}{p}}\left(\|Q^*\|_2^2+\|R^*\|_2^2\right).$$

Combining the above two equalities and noting that $s_c = \frac{n}{2} - \frac{1}{p}$ lead to identity (3.7).

Furthermore, using $\sigma_{p,n,\beta} = \left(\frac{2}{pn}\right)^{\frac{1}{2p}}\sqrt{\|Q^*\|_2^2+\|R^*\|_2^2}$, we obtain

$$\begin{aligned}\mathcal{V}[Q,R] &= \left(\|\nabla Q\|_2^2+\|\nabla R\|_2^2\right)^{\frac{s_c}{2}}\left(\|Q\|_2^2+\|R\|_2^2\right)^{\frac{1-s_c}{2}}\\ &= \lambda^{s_c-\frac{n}{2}}\sqrt{\|Q^*\|_2^2+\|R^*\|_2^2}\\ &= \lambda^{-\frac{1}{p}}\sqrt{\|Q^*\|_2^2+\|R^*\|_2^2} \equiv \sigma_{p,n,\beta}.\end{aligned}$$

Using lemma 1 and (3.4) yields

$$\begin{aligned}C_{n,p,\beta} &= \frac{p+1}{\left(\|Q^*\|_2^2+\|R^*\|_2^2\right)^p}\\ &= \frac{p+1}{(\mathcal{V}^{2p}[Q^*,R^*])^p}.\end{aligned}$$

Using the fact that $Q(x) = Q^*(\lambda x)$, $R(x) = R^*(\lambda x)$ and $\lambda = \sqrt{\frac{pn}{2}}$ derives (3.9). □

4. Invariant evolution flows

In this section, we shall give the invariant flows generated by the Cauchy problem (1.1) and (1.2).

We give the definition of two sets K_g and K_b.

Definition 3.

$$K_g := \{(\phi,\psi) \in H^1(R^n)\times H^1(R^n) : \mathcal{V}[\phi,\psi] < \mathcal{V}[Q,R], \wedge[\phi,\psi] < \wedge[Q,R]\},$$
$$K_b := \{(\phi,\psi) \in H^1(R^n)\times H^1(R^n) : \mathcal{V}[\phi,\psi] > \mathcal{V}[Q,R], \wedge[\phi,\psi] < \wedge[Q,R]\}.$$

Theorem 1. *Assume $\frac{2}{n} < p < A_n$. Then K_g and K_b are invariant under the flow generated by the Cauchy problem (1.1) and (1.2). More precisely, if $(\phi_0,\psi_0) \in K_g(K_b)$, then the corresponding solution $(\phi(t),\psi(t))$ satisfies $(\phi(t),\psi(t)) \in K_g(K_b)$.*

Proof. Let $(\phi_0,\psi_0) \in K_g$ and $(\phi(t),\psi(t))$ be the solution of the system (1.1) with the initial datum (ϕ_0,ψ_0). By the conservations of mass (1.3) and energy (1.4), one has

$$\wedge[\phi(t),\psi(t)] = \wedge[\phi_0,\psi_0] < \wedge[Q,R]. \tag{4.1}$$

To check that $(\phi(t),\psi(t)) \in K_g$, we only need to prove

$$\mathcal{V}[\phi(t),\psi(t)] < \mathcal{V}[Q,R], \qquad t \in [0,T). \tag{4.2}$$

If (4.2) is not true, because of $\mathcal{V}[\phi_0, \psi_0] < \mathcal{V}[Q, R]$, there would exist, by continuity, $t_1 \in [0, T)$ such that

$$\mathcal{V}(\phi(t_1), \psi(t_1)) = \mathcal{V}[Q, R], \qquad t \in [0, T). \tag{4.3}$$

However, it follows from proposition 1 and lemma 1 that

$$\begin{aligned}
\wedge^{\frac{1}{s_c}}[\phi(t_1), \psi(t_1)] &= E[\phi(t_1), \psi(t_1)] M^{\frac{1-s_c}{s_c}}[\phi(t_1), \psi(t_1)] \\
&= \frac{1}{2}\big(\|\nabla\phi(t_1)\|_2^2 + \|\nabla\psi(t_1)\|_2^2\big) M^{\frac{1-s_c}{s_c}}[\phi(t_1), \psi(t_1)] \\
&\quad - \frac{1}{2p+2}\big(\|\phi(t_1)\|_{2p+2}^{2p+2} + 2\beta\|\phi(t_1)\psi(t_1)\|_{p+1}^{p+1} + \|\psi(t_1)\|_{2p+2}^{2p+2}\big) \\
&\quad \times M^{\frac{1-s_c}{s_c}}[\phi(t_1), \psi(t_1)] \\
&\geqslant \frac{1}{2}\big(\|\nabla\phi(t_1)\|_2^2 + \|\nabla\psi(t_1)\|_2^2\big) M^{\frac{1-s_c}{s_c}}[\phi(t_1), \psi(t_1)] \\
&\quad - \frac{C_{n,p,\beta}}{2p+2}(M[\phi(t_1), \psi(t_1)])^{p+1-\frac{np}{2}}\big(\|\nabla\phi\|_2^2 + \|\nabla\psi(t_1)\|_2^2\big)^{\frac{np}{2}} \\
&\quad \times M^{\frac{1-s_c}{s_c}}[\phi(t_1), \psi(t_1)] \\
&= \frac{1}{2}\mathcal{V}^{\frac{2}{s_c}}[\phi(t_1), \psi(t_1)] - \frac{C_{n,p,\beta}}{2p+2}\mathcal{V}^{2(p+\frac{1}{s_c})}[\phi(t_1), \psi(t_1)].
\end{aligned} \tag{4.4}$$

Substituting (3.9) and (4.3) into (4.4) yields

$$\begin{aligned}
\wedge^{\frac{1}{s_c}}[\phi(t_1), \psi(t_1)] &\geqslant \frac{1}{2}\mathcal{V}^{\frac{2}{s_c}}[Q, R] - \frac{C_{n,p,\beta}}{2p+2}\mathcal{V}^{2(p+\frac{1}{s_c})}[Q, R] \\
&= \frac{1}{2}\mathcal{V}^{\frac{2}{s_c}}[Q, R] - \frac{1}{2p+2}\frac{p+1}{(\frac{np}{2})\mathcal{V}^{2p}[Q, R]}\mathcal{V}^{2\left(p+\frac{1}{s_c}\right)}[Q, R] \\
&= \frac{np-2}{2np}\mathcal{V}^{\frac{2}{s_c}}[Q, R] \\
&= \frac{s_c}{n}\sigma_{n,p,\beta}^{\frac{2}{s_c}}.
\end{aligned} \tag{4.5}$$

Combining (3.7) and (4.5), we get

$$\wedge[\phi(t_1), \psi(t_1)] \geqslant \left(\frac{s_c}{n}\right)^{s_c}\sigma_{n,p,\beta}^2 = \wedge[Q, R].$$

This violates $\wedge[\phi(t_1), \psi(t_1)] = \wedge[\phi_0, \psi_0] < \wedge[Q, R]$. Thus, inequality (4.2) is true. Hence, K_g is invariant under the flow generated by the Cauchy problem (1.1) and (1.2).

By the same argument as above, we can show that K_b is invariant under the flow generated by the Cauchy problem (1.1) and (1.2). This completes the proof of theorem 1. □

5. Blow-up threshold

To derive the results for the blow-up phenomena, we need to use the following variance $V(t)$ and two lemmas on the variance identity and uncertainty inequality respectively due to Fanelli and Montefusco [7] and Weinstein [19]:

$$V(t) = \int_{R^n} |x|^2(|\phi|^2 + |\psi|^2)\,\mathrm{d}x.$$

Lemma 3 (Fanelli and Montefusco [7])**.** *Let* (ϕ, ψ) *be a solution of the system (1.1); then the variance satisfies the variance identities*

$$V'(t) = 4\Im \int_{R^n} [(x \cdot \nabla\phi)\bar{\phi} + (x \cdot \nabla\psi)\bar{\psi}]\,\mathrm{d}x,$$

$$V''(t) = 8\int_{R^n} (|\nabla\phi|^2 + |\nabla\psi|^2)\,\mathrm{d}x - \frac{4np}{p+1}\int_{R^n} (|\phi|^{2p+2} + 2\beta|\phi|^{p+1}|\psi|^{p+1} + |\psi|^{2p+2})\,\mathrm{d}x.$$

Lemma 4 (Weinstein [19])**.** *For any* $u \in H^1$*, we have the uncertainty inequality*

$$\|u\|_2^2 \leqslant \frac{2}{n}\||x|u\|_2\|\nabla u\|_2.$$

The principal results are as follows.

Theorem 2. *Let* $\frac{2}{n} \leqslant p < A_n$ *and* $(|x|\phi_0, |x|\psi_0) \in L^2(R^n) \times L^2(R^n)$*. Assume that*

$$\wedge\,[\phi_0, \psi_0] < \wedge[Q, R] \equiv \left(\frac{s_c}{n}\right)^{s_c} (\sigma_{p,n,\beta})^2; \tag{5.1}$$

then the following two conclusions are valid.

(1) If $\mathcal{V}[\phi_0, \psi_0] < \mathcal{V}[Q, R]$*, then the solution exists globally in time.*
(2) If $\mathcal{V}[\phi_0, \psi_0] > \mathcal{V}[Q, R]$*, then the solution blows up in finite time.*

Remark 1. The above theorem gives a sharp threshold for the initial data which separates blow-up and global existence of a solution.

Proof.

(i) Let $\wedge[\phi_0, \psi_0] < \wedge[Q, R] \equiv \left(\frac{s_c}{n}\right)(\sigma_{p,n,\beta})^2$ and $\mathcal{V}[\phi(t), \psi(t)] < \mathcal{V}[Q, R]$, that is $(\phi_0, \psi_0) \in K_g$. Let $(\phi(t), \psi(t))$ be the corresponding solution of the Cauchy problem (1.1) and (1.2). It follows from theorem 1 that $(\phi(t), \psi(t)) \in K_g$. Hence,

$$\mathcal{V}[\phi(t), \psi(t)] := \left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right)^{\frac{s_c}{2}}\left(\|\phi\|_2^2 + \|\psi\|_2^2\right)^{\frac{1-s_c}{2}} < \sigma_{p,n,\beta}.$$

Using the above inequality and the mass conservation (1.3), we obtain that $(\phi(t), \psi(t))$ is bounded in $H^1(R^n) \times H^1(R^n)$. Therefore, by proposition 1, we know that the solution $(\phi(t), \psi(t))$ exists globally.

(ii) Suppose initially $\wedge[\phi_0, \psi_0] < \wedge[Q, R] \equiv \left(\frac{s_c}{n}\right)(\sigma_{p,n,\beta})^2$ and $\mathcal{V}[\phi_0, \psi_0] > \mathcal{V}[Q, R]$, that is $(\phi_0, \psi_0) \in K_b$. Let $(\phi(t), \psi(t))$ be the corresponding solution of the Cauchy problem (1.1) and (1.2). It follows from theorem 1 that $(\phi(t), \psi(t)) \in K_b$, which implies

$$\wedge[\phi(t), \psi(t)] < \wedge[Q, R] \equiv \left(\frac{s_c}{n}\right)(\sigma_{p,n,\beta})^2$$

and

$$\mathcal{V}[\phi(t), \psi(t)] > \mathcal{V}[Q, R].$$

Therefore, it follows from proposition 3 and the energy conservation (1.4) that

$$\begin{aligned}
V''(t)M^{\frac{1-s_c}{s_c}}[\phi(t), \psi(t)] &= \Bigg(8\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right) \\
&\quad - \frac{4np}{p+1}\left(\|\phi\|_{2p+2}^{2p+2} + 2\beta\|\phi\psi\|_{p+1}^{p+1} + \|\psi\|_{2p+2}^{2p+2}\right)\Bigg)M^{\frac{1-s_c}{s_c}}[\phi, \psi] \\
&= \left(8npE(\phi, \psi) - 4(np-2)\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right)\right)M^{\frac{1-s_c}{s_c}}[\phi, \psi] \\
&= 8np \wedge^{\frac{1}{s_c}}[\phi_0, \psi_0] - 4(np-2)\mathcal{V}^{\frac{2}{s_c}}[\phi, \psi] \\
&< 8np \wedge^{\frac{1}{s_c}}[\phi_0, \psi_0] - 4(np-2)\mathcal{V}^{\frac{2}{s_c}}[Q, R] \\
&= 8np \wedge^{\frac{1}{s_c}}[\phi_0, \psi_0] - 4(np-2)\sigma_{n,p,\beta}^{\frac{2}{s_c}}.
\end{aligned}$$

Noting (3.7), the above inequality implies that

$$V''(t)M^{\frac{1-s_c}{s_c}}[\phi(t),\psi(t)] < 8np\left(\wedge^{\frac{1}{s_c}}[\phi_0,\psi_0] - \wedge^{\frac{1}{s_c}}[Q,R]\right).$$

Thus, we get $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{R^n}|x|^2(|\phi|^2+|\psi|^2)\,\mathrm{d}x < -\delta < 0$. Here δ is the positive constant. Therefore, there exists a finite $T<\infty$ such that $\lim_{t\to T}V(t)=0$, which means that

$$\lim_{t\to T}\int_{R^n}|x|^2|\phi|^2\,\mathrm{d}x \to 0 \qquad \text{and} \qquad \lim_{t\to T}\int_{R^n}|x|^2|\psi|^2\,\mathrm{d}x \to 0.$$

Using lemma 3 and the mass identity (1.3), we get

$$\lim_{t\to T}\int_{R^n}|\nabla\phi|^2\,\mathrm{d}x \to \infty \qquad \text{and} \qquad \lim_{t\to T}\int_{R^n}|\nabla\psi|^2\,\mathrm{d}x \to \infty.$$

□

6. Concluding remarks

Remark 2. In theorem 2, if $p=\frac{2}{n}$, we have $s_c=\frac{n}{2}-\frac{1}{p}=0$. Thus, $K_g=\left\{(\phi_0,\psi_0): \|\phi_0\|_2^2+\|\psi_0\|_2^2 < \|Q\|_2^2+\|R\|_2^2\right\}$ and $K_b=\emptyset$. In this case, our theorem 2 coincides with the main result in [7]. Namely, our theorem 2 includes the main result presented in [7].

Remark 3. In the case of $p=1$ and $n=3$, the system (1.1) is physically relevant. In this case, theorem 2 can be stated as follows.

Suppose the initial data (ϕ_0,ψ_0) satisfy $M[\phi_0,\psi_0]E[\phi_0,\psi_0] < M[Q,R]E[Q,R]$.

- If $\left(\|\phi_0\|_2^2+\|\psi_0\|_2^2\right)\left(\|\nabla\phi_0\|_2^2+\|\nabla\psi_0\|_2^2\right) < \left(\|Q\|_2^2+\|R\|_2^2\right)\left(\|\nabla Q\|_2^2+\|\nabla R\|_2^2\right)$, then the solution $(\phi(t),\psi(t))$ globally exists.
- If $\left(\|\phi_0\|_2^2+\|\psi_0\|_2^2\right)\left(\|\nabla\phi_0\|_2^2+\|\nabla\psi_0\|_2^2\right) > \left(\|Q\|_2^2+\|R\|_2^2\right)\left(\|\nabla Q\|_2^2+\|\nabla R\|_2^2\right)$, then the solution $(\phi(t),\psi(t))$ blows up in finite time.

If the energy is negative, then via the Gagliardo–Nirenberg inequality (3.6) we automatically have $\left(\|\phi_0\|_2^2+\|\psi_0\|_2^2\right)\left(\|\nabla\phi_0\|_2^2+\|\nabla\psi_0\|_2^2\right) > \left(\|Q\|_2^2+\|R\|_2^2\right)\left(\|\nabla Q\|_2^2+\|\nabla R\|_2^2\right)$ and the second of the above two cases is valid.

To interpret the physical implication of the above results, we note from (1.3) and (1.4) that $M(t)=M(\phi(t),\psi(t)) := \|\phi_0\|_2^2+\|\psi_0\|_2^2$ is the mass of the system, and $E(t)=E(\phi(t),\psi(t))=T(t)+U(t)$ denotes the total energy of the system, where $T(t)=T(\phi(t),\psi(t)) := \frac{1}{2}\left(\|\nabla\phi(t)\|_2^2+\|\nabla\psi(t)\|_2^2\right)$ corresponds to the kinetic energy of the system while $U(t)=U(\phi(t),\psi(t)) := -\frac{1}{2p+2}\left(\|\phi\|_{2p+2}^{2p+2}+2\beta\|\phi\psi\|_{p+1}^{p+1}+\|\psi\|_{2p+2}^{2p+2}\right)$ corresponds to the potential energy of the system. Thus, in physics terms, our results show that if $M(\phi_0,\psi_0)E(\phi_0,\psi_0) < M(Q,R)E(Q,R)$, the solution $(\phi(t),\psi(t))$ blows up in finite time when $M(\phi_0,\psi_0)T(\phi_0,\psi_0) > M(Q,R)T(Q,R)$, and the solution exists globally when $M(\phi_0,\psi_0)T(\phi_0,\psi_0) < M(Q,R)T(Q,R)$. This result suggests that the product of the mass and the kinetic energy plays an important role in the evolution of the system.

Remark 4. The conclusion in theorem 2 can be expressed equivalently as follows.

Assume $\frac{2}{n}\leqslant p < A_n$ and

$$0 < E(\phi_0,\psi_0) < \frac{s_c}{n}M^{\frac{1-s_c}{s_c}}\sigma_{n,p,\beta}^{\frac{2}{s_c}}.$$

Then one has the following results.

(1) If $\|\nabla\phi_0\|_2^2+\|\nabla\psi_0\|_2^2 < \left(\frac{\|Q\|_2^2+\|R\|_2^2}{\|\phi_0\|_2^2+\|\psi_0\|_2^2}\right)^{\frac{1-s_c}{s_c}}\left(\|\nabla Q\|_2^2+\|\nabla R\|_2^2\right)$, then $(\phi(t),\psi(t))$ globally exists in $H^1(R^n)\times H^1(R^n)$;

(2) If $\|\nabla\phi_0\|_2^2 + \|\nabla\psi_0\|_2^2 > \left(\frac{\|Q\|_2^2+\|R\|_2^2}{\|\phi_0\|_2^2+\|\psi_0\|_2^2}\right)^{\frac{1-s_c}{s_c}}\left(\|\nabla Q\|_2^2 + \|\nabla R\|_2^2\right)$, then $(\phi(t), \psi(t))$ blows up in a finite time.

Remark 5. From [7] and the conclusion in theorem 2, we know that

- If $E(\phi_0, \psi_0) < 0$, $(\phi(t), \psi(t))$ blows up in finite time.
- If $0 < E(\phi_0, \psi_0) < \frac{s_c}{n} M^{\frac{1-s_c}{s_c}} \sigma_{n,p,\beta}^{\frac{2}{s_c}}$ and $\|\nabla\phi_0\|_2^2 + \|\nabla\psi_0\|_2^2 < \left(\frac{\|Q\|_2^2+\|R\|_2^2}{\|\phi_0\|_2^2+\|\psi_0\|_2^2}\right)^{\frac{1-s_c}{s_c}}\left(\|\nabla Q\|_2^2 + \|\nabla R\|_2^2\right)$, then $(\phi(t), \psi(t))$ globally exists in $H^1(R^n) \times H^1(R^n)$.
- If $0 < E(\phi_0, \psi_0) < \frac{s_c}{n} M^{\frac{1-s_c}{s_c}} \sigma_{n,p,\beta}^{\frac{2}{s_c}}$ and $\|\nabla\phi_0\|_2^2 + \|\nabla\psi_0\|_2^2 > \left(\frac{\|Q\|_2^2+\|R\|_2^2}{\|\phi_0\|_2^2+\|\psi_0\|_2^2}\right)^{\frac{1-s_c}{s_c}}\left(\|\nabla Q\|_2^2 + \|\nabla R\|_2^2\right)$, then $(\phi(t), \psi(t))$ blows up in a finite time.

Remark 6. Parts (i) and (ii) in proposition 2 imply that if $\beta < 2^p - 1$, then any ground state of the elliptic system (1.5) is a scalar function, namely one of the components of the ground-state solution is zero. So we can assume, without loss of generality, that the ground-state solution is $(z, 0)$, where $z \in H^1$ is the unique ground-state solution (see [11, 18]):

$$\triangle z - \frac{(2-n)p-2}{2} z + |z|^{2p} z = 0.$$

This implies that the constant $\sigma_{n,p,\beta} = \sigma_{n,p}$ depends only on n and p for any $\beta < 2^p - 1$.

In the case of $\beta > 2^p - 1$, $\sigma_{n,p,\beta} = \sigma_{n,p}$ depends on n, p and β. But we can estimate it using a suitable test pair. Let $\hat{z}$ be the unique positive ground-state solution of

$$\triangle \hat{z} - \frac{(2-n)p-2}{2} \hat{z} + (1+\beta)|\hat{z}|^{2p} \hat{z} = 0.$$

It is easy to see that the pair $(\hat{z}, \hat{z})$ is a positive solution of (1.5) for any β, and the following inequality holds:

$$\begin{aligned}\sigma_{n,p,\beta} &= \mathcal{V}[Q,R] = \left(\|\nabla Q\|_2^2 + \|\nabla R\|_2^2\right)^{\frac{s_c}{2}}\left(\|Q\|_2^2 + \|R\|_2^2\right)^{\frac{1-s_c}{2}} \\ &\leqslant \left(\|\nabla \hat{z}\|_2^2 + \|\nabla \hat{z}\|_2^2\right)^{\frac{s_c}{2}}\left(\|\hat{z}\|_2^2 + \|\hat{z}\|_2^2\right)^{\frac{1-s_c}{2}} \\ &= 2\|\nabla \hat{z}\|_2^{s_c}\|\hat{z}\|_2^{1-s_c}.\end{aligned}$$

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